# Calcul des déformations des barres élastiques

## Barres élastiques en arc de cercle - Modèle encastré - libre

### Torseur des forces de cohésion

$$\mathbf{r_F}(\psi_F) := \begin{pmatrix} R\cdot\cos(\psi_F) \\ R\cdot\sin(\psi_F) \\ 0 \end{pmatrix} \qquad \mathbf{r_S}(\alpha') := \begin{pmatrix} R\cdot\cos(\alpha') \\ R\cdot\sin(\alpha') \\ 0 \end{pmatrix} \qquad \mathbf{r_v}(\alpha) := \begin{pmatrix} R\cdot\cos(\alpha) \\ R\cdot\sin(\alpha) \\ 0 \end{pmatrix} \qquad \mathbf{r_q}(\chi) := \begin{pmatrix} R\cdot\cos(\chi) \\ R\cdot\sin(\chi) \\ 0\cdot m \end{pmatrix}$$

### Forces et couples concentrés

$$\mathbf{M_c}(\psi_F, \alpha') := \left[\mathbf{C} + (\mathbf{r_F}(\psi_F) - \mathbf{r_S}(\alpha')) \times \mathbf{F}\right]\cdot(\alpha' \le \psi_F)$$

$$\mathbf{M_c}(\psi_F, \alpha') := \left[\begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix} + R\cdot\begin{bmatrix} F_z\cdot(\sin(\psi_F) - \sin(\alpha')) \\ -F_z\cdot(\cos(\psi_F) - \cos(\alpha')) \\ -F_x\cdot(\sin(\psi_F) - \sin(\alpha')) + F_y\cdot(\cos(\psi_F) - \cos(\alpha')) \end{bmatrix}\right]\cdot(\alpha' \le \psi_F)$$

### Forces distribuées

$$\mathbf{q}(\chi) := \begin{pmatrix} q_x(\chi) & q_y(\chi) & q_z(\chi) \end{pmatrix}^T \qquad \mathbf{M_q}(\psi_q, \alpha') := \left[\int_{\alpha'}^{\psi_q} (\mathbf{r_q}(\chi) - \mathbf{r_S}(\alpha')) \times \mathbf{q}(\chi)\cdot R\, d\chi\right]\cdot(\alpha' \le \psi_q)$$

$$\mathbf{M_q}(\psi_q, \alpha') := R^2\cdot\begin{bmatrix} \displaystyle\int_{\alpha'}^{\psi_q} q_z(\chi)\cdot(\sin(\chi) - \sin(\alpha'))\, d\chi \\ \displaystyle\int_{\alpha'}^{\psi_q} -q_z(\chi)\cdot(\cos(\chi) - \cos(\alpha'))\, d\chi \\ \displaystyle\int_{\alpha'}^{\psi_q} -q_x(\chi)\cdot(\sin(\chi) - \sin(\alpha'))\, d\chi + \int_{\alpha'}^{\psi_q} q_y(\chi)\cdot(\cos(\chi) - \cos(\alpha'))\, d\chi \end{bmatrix}\cdot(\alpha' \le \psi_q)$$

Torseur $\qquad \mathbf{M_{cq}}(\psi_F, \psi_q, \alpha') := \mathbf{M_c}(\psi_F, \alpha') + \mathbf{M_q}(\psi_q, \alpha')$

### Sollicitations

$$\mathbf{e'_1}(\alpha') := \begin{pmatrix} -\sin(\alpha') & \cos(\alpha') & 0 \end{pmatrix}^T \qquad \mathbf{e'_2}(\alpha') := \begin{pmatrix} -\cos(\alpha') & -\sin(\alpha') & 0 \end{pmatrix}^T \qquad \mathbf{e'_3}(\alpha') := \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$$

Moments de torsion

$$M_t(\psi_F, \psi_q, \alpha') := \mathbf{M_{cq}}(\psi_F, \psi_q, \alpha')\cdot\mathbf{e'_1}(\alpha')$$

$$Mc_t(\psi_F, \alpha') := \mathbf{M_c}(\psi_F, \alpha')\cdot\mathbf{e'_1}(\alpha') \qquad Mq_t(\psi_q, \alpha') := \mathbf{M_q}(\psi_q, \alpha')\cdot\mathbf{e'_1}(\alpha')$$

Moments de flexion

$$M_{f2}(\psi_F, \psi_q, \alpha') := \mathbf{M_{cq}}(\psi_F, \psi_q, \alpha')\cdot\mathbf{e'_2}(\alpha')$$

$$M_{f3}(\psi_F, \psi_q, \alpha') := \mathbf{M_{cq}}(\psi_F, \psi_q, \alpha')\cdot\mathbf{e'_3}(\alpha')$$

$$Mc_{f2}(\psi_F, \alpha') := \mathbf{M_c}(\psi_F, \alpha')\cdot\mathbf{e'_2}(\alpha') \qquad Mq_{f2}(\psi_q, \alpha') := \mathbf{M_q}(\psi_q, \alpha')\cdot\mathbf{e'_2}(\alpha')$$

$$Mc_{f3}(\psi_F, \alpha') := \mathbf{M_c}(\psi_F, \alpha')\cdot\mathbf{e'_3}(\alpha') \qquad Mq_{f3}(\psi_q, \alpha') := \mathbf{M_q}(\psi_q, \alpha')\cdot\mathbf{e'_3}(\alpha')$$

## *Contraintes*

$k$ = limite d'élasticité en traction / limite d'élasticité en compression

M point de la section droite le plus éloigné de O sur l'axe O$\mathbf{e'_2}$

N point de la section droite le plus éloigné de O sur l'axe O$\mathbf{e'_3}$

$$\tau_M(\psi_F, \psi_q, \alpha') := \frac{M_t(\psi_F, \psi_q, \alpha')}{W_t} \qquad \tau_N(\psi_F, \psi_q, \alpha') := \frac{M_t(\psi_F, \psi_q, \alpha')}{W'_t}$$

$$\sigma_M(\psi_F, \psi_q, \alpha') := \frac{M_{f3}(\psi_F, \psi_q, \alpha')}{W_{f3}} \qquad \sigma_N(\psi_F, \psi_q, \alpha') := \frac{M_{f2}(\psi_F, \psi_q, \alpha')}{W_{f2}}$$

$$\sigma_{équiv\_M}(k, \psi_F, \psi_q, \alpha') := \frac{1-k}{2}\cdot\left|\sigma_M(\psi_F, \psi_q, \alpha')\right| + \frac{1+k}{2}\cdot\sqrt{\sigma_M(\psi_F, \psi_q, \alpha')^2 + 4\cdot\tau_M(\psi_F, \psi_q, \alpha')^2}$$

$$\sigma_{équiv\_N}(k, \psi_F, \psi_q, \alpha') := \frac{1-k}{2}\cdot\left|\sigma_N(\psi_F, \psi_q, \alpha')\right| + \frac{1+k}{2}\cdot\sqrt{\sigma_N(\psi_F, \psi_q, \alpha')^2 + 4\cdot\tau_N(\psi_F, \psi_q, \alpha')^2}$$

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## *Calcul des déplacements par les intégrales de Mohr*

### Calcul des déplacements linéiques

Force unitaire virtuelle $\quad \mathbf{v}(\lambda, \gamma) := (\cos(\lambda)\cdot\sin(\gamma) \quad \sin(\lambda)\cdot\sin(\gamma) \quad \cos(\gamma))^T$

Sollicitations dues à la force unitaire $\quad \mathbf{M_v}(\alpha, \lambda, \gamma, \alpha') := \left[\left(\mathbf{r_v}(\alpha) - \mathbf{r_S}(\alpha')\right)\times \mathbf{v}(\lambda, \gamma)\right]\cdot(\alpha' < \alpha)$

$$\mathbf{M_v}(\alpha, \lambda, \gamma, \alpha') := R\cdot\begin{bmatrix} \cos(\gamma)\cdot(\sin(\alpha) - \sin(\alpha')) \\ -\cos(\gamma)\cdot(\cos(\alpha) - \cos(\alpha')) \\ -(\cos(\lambda)\cdot\sin(\gamma))\cdot(\sin(\alpha) - \sin(\alpha')) + \sin(\lambda)\cdot\sin(\gamma)\cdot(\cos(\alpha) - \cos(\alpha')) \end{bmatrix}\cdot(\alpha' < \alpha)$$

$$M_{tv}(\alpha, \lambda, \gamma, \alpha') := \mathbf{M_v}(\alpha, \lambda, \gamma, \alpha')\cdot\mathbf{e'_1}(\alpha') \qquad lim(\alpha, \psi) := \alpha\cdot(\alpha \le \psi) + \psi\cdot(\alpha > \psi)$$

$$M_{fv2}(\alpha, \lambda, \gamma, \alpha') := \mathbf{M_v}(\alpha, \lambda, \gamma, \alpha')\cdot\mathbf{e'_2}(\alpha') \qquad M_{fv3}(\alpha, \lambda, \gamma, \alpha') := \mathbf{M_v}(\alpha, \lambda, \gamma, \alpha')\cdot\mathbf{e'_3}(\alpha')$$

Déplacement dans la direction de **v**

$$\delta c_{tv}(\psi_F, \alpha, \lambda, \gamma) := \frac{R}{G\cdot J_t}\cdot\int_0^{lim(\alpha, \psi_F)} Mc_t(\psi_F, \alpha')\cdot M_{tv}(\alpha, \lambda, \gamma, \alpha')\, d\alpha'$$

$$\delta q_{tv}(\psi_q, \alpha, \lambda, \gamma) := \frac{R}{G\cdot J_t}\cdot\int_0^{lim(\alpha, \psi_q)} Mq_t(\psi_q, \alpha')\cdot M_{tv}(\alpha, \lambda, \gamma, \alpha')\, d\alpha'$$

$$\delta_{tv}(\psi_F, \psi_q, \alpha, \lambda, \gamma) := \delta c_{tv}(\psi_F, \alpha, \lambda, \gamma) + \delta q_{tv}(\psi_q, \alpha, \lambda, \gamma)$$

$$\delta c_{fv2}(\psi_F, \alpha, \lambda, \gamma) := \frac{R}{E\cdot I_{22}}\cdot\int_0^{lim(\alpha, \psi_F)} Mc_{f2}(\psi_F, \alpha')\cdot M_{fv2}(\alpha, \lambda, \gamma, \alpha')\, d\alpha'$$

$$\delta q_{fv2}(\psi_q, \alpha, \lambda, \gamma) := \frac{R}{E\cdot I_{22}}\cdot\int_0^{lim(\alpha, \psi_q)} Mq_{f2}(\psi_q, \alpha')\cdot M_{fv2}(\alpha, \lambda, \gamma, \alpha')\, d\alpha'$$

$$\delta_{fv2}(\psi_F, \psi_q, \alpha, \lambda, \gamma) := \delta c_{fv2}(\psi_F, \alpha, \lambda, \gamma) + \delta q_{fv2}(\psi_q, \alpha, \lambda, \gamma)$$

$$\delta c_{fv3}(\psi_F, \alpha, \lambda, \gamma) := \frac{R}{E \cdot I_{33}} \cdot \int_0^{lim(\alpha, \psi_F)} Mc_{f3}(\psi_F, \alpha') \cdot M_{fv3}(\alpha, \lambda, \gamma, \alpha')\, d\alpha'$$

$$\delta q_{fv3}(\psi_q, \alpha, \lambda, \gamma) := \frac{R}{E \cdot I_{33}} \cdot \int_0^{lim(\alpha, \psi_q)} Mq_{f3}(\psi_q, \alpha') \cdot M_{fv3}(\alpha, \lambda, \gamma, \alpha')\, d\alpha'$$

$$\delta_{fv3}(\psi_F, \psi_q, \alpha, \lambda, \gamma) := \delta c_{fv3}(\psi_F, \alpha, \lambda, \gamma) + \delta q_{fv3}(\psi_q, \alpha, \lambda, \gamma)$$

$$\delta_v(\psi_F, \psi_q, \alpha, \lambda, \gamma) := \delta_{tv}(\psi_F, \psi_q, \alpha, \lambda, \gamma) + \delta_{fv2}(\psi_F, \psi_q, \alpha, \lambda, \gamma) + \delta_{fv3}(\psi_F, \psi_q, \alpha, \lambda, \gamma)$$

## Calcul des déplacements angulaires

Couple unitaire virtuel $\quad \mathbf{cv}(\lambda_c, \gamma_c) := \begin{pmatrix} cos(\lambda_c) \cdot sin(\gamma_c) & sin(\lambda_c) \cdot sin(\gamma_c) & cos(\gamma_c) \end{pmatrix}^T$

Sollicitations dues au couple unitaire $\quad \mathbf{M_{cv}}(\alpha, \lambda_c, \gamma_c, \alpha') := \mathbf{cv}(\lambda_c, \gamma_c) \cdot (\alpha' < \alpha)$

$$\mathbf{M_{cv}}(\alpha, \lambda_c, \gamma_c, \alpha') := \begin{pmatrix} cos(\lambda_c) \cdot sin(\gamma_c) \\ sin(\lambda_c) \cdot sin(\gamma_c) \\ cos(\gamma_c) \end{pmatrix} \cdot (\alpha' < \alpha)$$

$$M_{tcv}(\alpha, \lambda_c, \gamma_c, \alpha') := \mathbf{M_{cv}}(\alpha, \lambda_c, \gamma_c, \alpha') \cdot \mathbf{e'_1}(\alpha')$$

$$M_{fcv2}(\alpha, \lambda_c, \gamma_c, \alpha') := \mathbf{M_{cv}}(\alpha, \lambda_c, \gamma_c, \alpha') \cdot \mathbf{e'_2}(\alpha') \qquad M_{fcv3}(\alpha, \lambda_c, \gamma_c, \alpha') := \mathbf{M_{cv}}(\alpha, \lambda_c, \gamma_c, \alpha') \cdot \mathbf{e'_3}(\alpha')$$

Déplacement angulaire autour de l'axe défini par **cv**

$$\theta c_{tcv}(\psi_F, \alpha, \lambda_c, \gamma_c) := \frac{R}{G \cdot J_t} \cdot \int_0^{lim(\alpha, \psi_F)} Mc_t(\psi_F, \alpha') \cdot M_{tcv}(\alpha, \lambda_c, \gamma_c, \alpha')\, d\alpha'$$

$$\theta q_{tcv}(\psi_q, \alpha, \lambda_c, \gamma_c) := \frac{R}{G \cdot J_t} \cdot \int_0^{lim(\alpha, \psi_q)} Mq_t(\psi_q, \alpha') \cdot M_{tcv}(\alpha, \lambda_c, \gamma_c, \alpha')\, d\alpha'$$

$$\theta_{tcv}(\psi_F, \psi_q, \alpha, \lambda_c, \gamma_c) := \theta c_{tcv}(\psi_F, \alpha, \lambda_c, \gamma_c) + \theta q_{tcv}(\psi_q, \alpha, \lambda_c, \gamma_c)$$

$$\theta c_{fcv2}(\psi_F, \alpha, \lambda_c, \gamma_c) := \frac{R}{E \cdot I_{22}} \cdot \int_0^{lim(\alpha, \psi_F)} Mc_{f2}(\psi_F, \alpha') \cdot M_{fcv2}(\alpha, \lambda_c, \gamma_c, \alpha')\, d\alpha'$$

$$\theta q_{fcv2}(\psi_q, \alpha, \lambda_c, \gamma_c) := \frac{R}{E \cdot I_{22}} \cdot \int_0^{lim(\alpha, \psi_q)} Mq_{f2}(\psi_q, \alpha') \cdot M_{fcv2}(\alpha, \lambda_c, \gamma_c, \alpha')\, d\alpha'$$

$$\theta_{fcv2}(\psi_F, \psi_q, \alpha, \lambda_c, \gamma_c) := \theta c_{fcv2}(\psi_F, \alpha, \lambda_c, \gamma_c) + \theta q_{fcv2}(\psi_q, \alpha, \lambda_c, \gamma_c)$$

$$\theta c_{fcv3}(\psi_F, \alpha, \lambda_c, \gamma_c) := \frac{R}{E \cdot I_{33}} \cdot \int_0^{lim(\alpha, \psi_F)} Mc_{f3}(\psi_F, \alpha') \cdot M_{fcv3}(\alpha, \lambda_c, \gamma_c, \alpha')\, d\alpha'$$

$$\theta q_{fcv3}(\psi_q, \alpha, \lambda_c, \gamma_c) := \frac{R}{E \cdot I_{33}} \cdot \int_0^{lim(\alpha, \psi_q)} Mq_{f3}(\psi_q, \alpha') \cdot M_{fcv3}(\alpha, \lambda_c, \gamma_c, \alpha')\, d\alpha'$$

$$\theta_{fcv3}(\psi_F, \psi_q, \alpha, \lambda_c, \gamma_c) := \theta c_{fcv3}(\psi_F, \alpha, \lambda_c, \gamma_c) + \theta q_{fcv3}(\psi_q, \alpha, \lambda_c, \gamma_c)$$

$$\theta_{cv}(\psi_F, \psi_q, \alpha, \lambda_c, \gamma_c) := \theta_{tcv}(\psi_F, \psi_q, \alpha, \lambda_c, \gamma_c) + \theta_{fcv2}(\psi_F, \psi_q, \alpha, \lambda_c, \gamma_c) + \theta_{fcv3}(\psi_F, \psi_q, \alpha, \lambda_c, \gamma_c)$$